

# SPDES LAW EQUIVALENCE AND THE COMPACT SUPPORT PROPERTY: APPLICATIONS TO THE ALLEN-CAHN SPDE

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**ABSTRACT.** Using our uniqueness in law transfer result for SPDEs, described in a recent note, we prove the equivalence of laws of SPDEs differing by a drift, under vastly applicable conditions. This gives us the equivalence in the compact support property among a large class of SPDEs. As an important application, we prove the equivalence in law of the Allen-Cahn and the associated heat SPDEs; and we give a criterion for the compact support property to hold for the Allen-Cahn SPDE with diffusion function  $a(t, x, u) = Cu^\gamma$ , with  $C \neq 0$  and  $1/2 \leq \gamma < 1$ .

## 1. STATEMENTS AND DISCUSSIONS OF RESULTS.

We start by considering the pair of parabolic SPDEs

$$(1.1) \quad \begin{cases} \frac{\partial U}{\partial t} = \Delta_x U + b(t, x, U) + a(t, x, U) \frac{\partial^2 W}{\partial t \partial x}; & (t, x) \in \widetilde{\mathcal{R}}_T, \\ U_x(t, -\infty) = U_x(t, \infty) = 0; & 0 < t \leq T, \\ U(0, x) = h(x); & x \in \mathbb{R}, \end{cases}$$

and

$$(1.2) \quad \begin{cases} \frac{\partial V}{\partial t} = \Delta_x V + (b + d)(t, x, V) + a(t, x, V) \frac{\partial^2 W}{\partial t \partial x}; & (t, x) \in \widetilde{\mathcal{R}}_T, \\ V_x(t, -\infty) = V_x(t, \infty) = 0; & 0 < t \leq T, \\ V(0, x) = h(x); & x \in \mathbb{R}. \end{cases}$$

on  $\mathcal{R}_T \triangleq [0, T] \times \mathbb{R}$ , where  $W(t, x)$  is the Brownian sheet corresponding to the driving space-time white noise, written formally as  $\partial^2 W / \partial t \partial x$ . As in Walsh [15], white noise is regarded as a continuous orthogonal martingale measure, which we denote by  $\mathcal{W}$ . The diffusion  $a(t, x, u)$  and the drifts  $b(t, x, u)$  and  $d(t, x, u)$  are Borel-measurable  $\mathbb{R}$ -valued functions on  $\mathcal{R}_T \times \mathbb{R}$ ; and  $h : \mathbb{R} \rightarrow \mathbb{R}$  is a bounded continuous function. Henceforth, we will denote (1.1) and (1.2) by  $e_{heat}^{Neu}(a, b, h)$  and  $e_{heat}^{Neu}(a, b + d, h)$ , respectively. When  $b \equiv 0$ , we denote (1.1) by  $e_{heat}^{Neu}(a, 0, h)$ . In the interest of getting quickly to our main results, we refer the reader to [5] for the rigorous interpretation of all SPDEs considered in this paper, with the obvious modifications to accomodate the change of space from  $\mathcal{R}_{T,L} = [0, T] \times [0, L]$  to  $\mathcal{R}_T$ . Also, the law of a random variable  $X$  under the probability measure  $\mathbb{P}$  is denoted by  $\mathbb{L}_{\mathbb{P}}^X$ . Proceeding toward a precise statement of our results, let  $R_u(t, x) \triangleq d(t, x, u)/a(t, x, u)$ , for any  $(t, x, u) \in \mathcal{R}_T \times \mathbb{R}$ , whenever the ratio is well defined. Let  $\lambda$  denote Lebesgue measure. Our law equivalence result for the pair  $e_{heat}^{Neu}(a, b, h)$  and  $e_{heat}^{Neu}(a, b + d, h)$  can now be stated as

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**Theorem 1.1.** *Let  $(V, W^{(1)})$  be a solution (weak or strong) to  $e_{heat}^{Neu}(a, b + d, h)$  on some probability space  $(\Omega^{(1)}, \mathcal{H}, \{\mathcal{H}_t\}, \mathbb{Q})$ . Assume that  $R_U$  and  $R_V$  are in  $L^2(\mathcal{R}_T, \lambda)$ , almost surely, whenever the random fields  $U$  and  $V$  solve (weakly or strongly)  $e_{heat}^{Neu}(a, b, h)$  and  $e_{heat}^{Neu}(a, b + d, h)$ , respectively. Assume further that there is a unique-in-law solution  $(U, W^{(2)})$  to the heat SPDE  $e_{heat}(a, b, h)$  on  $(\Omega^{(2)}, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ . Then  $\mathbb{L}_{\mathbb{Q}}^V$  and  $\mathbb{L}_{\mathbb{P}}^U$  are mutually absolutely continuous (on  $(C(\mathcal{R}_T; \mathbb{R}))$ ).*

**Remark 1.1.** *By Theorem 1.1 in [5], which can trivially be extended from  $\mathcal{R}_{T,L}$  to  $\mathcal{R}_T$  (replacing  $[0, L]$  with  $\mathbb{R}$  in the  $L^2$  condition and in the proof), uniqueness in law for the SPDE  $e_{heat}(a, b, h)$  is equivalent to uniqueness in law for  $e_{heat}(a, b + d, h)$  under the  $L^2(\mathcal{R}_T, \lambda)$  condition on  $R_U$  and  $R_V$ . So, we can replace the uniqueness assumption on  $e_{heat}(a, b, h)$  in Theorem 1.1 above by that on  $e_{heat}(a, b + d, h)$ . Also, our Neumann conditions may be replaced by Dirichlet conditions without affecting the conclusion of Theorem 1.1. Finally, Theorem 1.1 and its proof are valid when  $\mathcal{R}_T$  is replaced by  $\mathcal{R}_{T,L}$  (replacing  $\mathbb{R}$  with  $[0, L]$ ).*

As an immediate consequence of Theorem 1.1 and Remark 1.1, we get the following law equivalence between the Allen-Cahn SPDE

$$(1.3) \quad \begin{cases} \frac{\partial V}{\partial t} = \Delta_x V + 2V(1 - V^2) + CV^\gamma \frac{\partial^2 W}{\partial t \partial x}; & (t, x) \in \widetilde{\mathcal{R}}_T, \ C > 0, \\ V_x(t, -\infty) = V_x(t, \infty) = 0; & 0 < t \leq T, \\ V(0, x) = h(x); & x \in \mathbb{R}, \end{cases}$$

and its associated heat SPDE (the one obtained from the Allen-Cahn SPDE (1.3) by removing the Allen-Cahn nonlinearity). We note here that the proof of the uniqueness for the Allen-Cahn SPDE in Theorem 1.2 in [5] works just as well for the case  $\gamma = 1/2$ , in addition to  $\frac{1}{2} < \gamma < 1$ , because the SPDE in (1.1) with  $b \equiv 0$  and  $a(t, x, u) = Cu^{1/2}$  admits uniqueness in law as discussed in [12] p. 326 and in [14].

**Corollary 1.1.** *Suppose that  $V$  and  $U$  solve (weakly or strongly) the Allen-Cahn SPDE and its associated heat SPDE, respectively, on  $\mathcal{R}_{T,L}$  (see (0.3) in [5]) and with  $1/2 \leq \gamma \leq 1$ . Then the laws of  $U$  and  $V$  are equivalent (on  $(C(\mathcal{R}_{T,L}; \mathbb{R}))$ ). If  $\mathcal{R}_{T,L}$  is replaced with  $\mathcal{R}_T$ , if  $1/2 \leq \gamma < 1$ , if  $h(x)$  has compact support, and if  $R_V$  is in  $L^2(\mathcal{R}_T, \lambda)$  a.s.:*

$$(1.4) \quad \int_{\mathcal{R}_T} R_V^2(t, x) dt dx = \frac{4}{C^2} \int_{\mathcal{R}_T} V^{2(1-\gamma)} (V^4 - 2V^2 + 1) dt dx < \infty; \text{ almost surely,}$$

*then the laws of  $U$  and  $V$  are equivalent (on  $(C(\mathcal{R}_T; \mathbb{R}))$ ).*

**Remark 1.2.** *In the first part of Corollary 1.1, the continuity of  $U$  and  $V$  insures that  $R_U$  and  $R_V$  are in  $L^2(\mathcal{R}_{T,L}, \lambda)$ , for  $0 \leq \gamma \leq 1$  (see the proof of Theorem 1.2 in [5]). When  $\mathcal{R}_{T,L}$  is replaced by  $\mathcal{R}_T$ , we do not require that  $R_U$  be in  $L^2(\mathcal{R}_T, \lambda)$  ((1.4) with  $U$  instead of  $V$ ). This is because  $R_U$  is already in  $L^2(\mathcal{R}_T, \lambda)$ , since  $U(t, \cdot)$  has compact support for each  $t$  in the range  $1/2 \leq \gamma < 1$  by [12, 10]. Also, when we replace  $\mathcal{R}_{T,L}$  with  $\mathcal{R}_T$ ,  $\gamma \leq 1$  is replaced with  $\gamma < 1$ ; since, in this case, when  $\gamma = 1$  the integrability assumption in (1.4) has obvious problems for both  $U$  and  $V$ .*

Let  $\Omega = C(\mathcal{R}_T; \mathbb{R})$  and denote elements of  $\Omega$  by  $\omega$ . Let  $X$  be the coordinate mapping process on  $\Omega$ :  $X_\omega(t, x) \triangleq \omega(t, x)$ . Denote by  $\mathcal{G}_{t,x}^X$ ,  $\mathcal{G}_{t,\cdot}^X$ , and  $\mathcal{G}_{\cdot,\cdot}^X$  the sigma

fields of subsets of  $\Omega$  generated by  $X$  when  $(t, x)$  is fixed, when  $t$  is fixed but  $x$  is not, and when both  $t$  and  $x$  are not fixed, respectively. I.e.,

$$\begin{aligned}\mathcal{G}_{t,x}^X &= \sigma\left(\left\{\omega \in \Omega; X_\omega(t, x) = \omega(t, x) \in A\right\}; A \in \mathcal{B}(\mathbb{R})\right); \quad (t, x) \in \mathcal{R}_T, \\ \mathcal{G}_{t,\cdot}^X &= \sigma\left(\left\{\omega \in \Omega; (X_\omega(t, x_1) = \omega(t, x_1), \dots, X_\omega(t, x_n) = \omega(t, x_n)) \in A\right\}; \right. \\ &\quad \left. n \geq 1, A \in \mathcal{B}(\mathbb{R}^n), x_i \in \mathbb{R}, i = 1, \dots, n\right); \quad t \in [0, T], \\ \mathcal{G}_{\cdot,\cdot}^X &= \sigma\left(\left\{\omega \in \Omega; (X_\omega(t_1, x_1) = \omega(t_1, x_1), \dots, X_\omega(t_n, x_n) = \omega(t_n, x_n)) \in A\right\}; \right. \\ &\quad \left. n \geq 1, A \in \mathcal{B}(\mathbb{R}^n), (t_i, x_i) \in \mathcal{R}_T, i = 1, \dots, n\right).\end{aligned}$$

Then, clearly,  $\mathcal{G}_{t,x}^X \subseteq \mathcal{G}_{t,\cdot}^X \subseteq \mathcal{B}(C(\mathcal{R}_T; \mathbb{R})) = \mathcal{G}_{\cdot,\cdot}^X$ , the last equality is a trivial extension of Problem 4.2 p. 60 in [9], and so absolute continuity on  $\mathcal{B}(C(\mathcal{R}_T; \mathbb{R}))$  implies absolute continuity on  $\mathcal{G}_{t,x}^X$  and  $\mathcal{G}_{t,\cdot}^X$ . This observation along with Theorem 1.1 easily give us

**Corollary 1.2.** *Under the conditions of Theorem 1.1,  $\mathbb{L}_{\mathbb{Q}}^{V(t,x)}$  is equivalent to  $\mathbb{L}_{\mathbb{P}}^{U(t,x)}$  on  $\mathbb{R}$ , for every  $(t, x) \in \mathcal{R}_T$  (in particular, if one is absolutely continuous with respect to Lebesgue measure then so is the other); and  $\mathbb{L}_{\mathbb{Q}}^{V(t,\cdot)}$  is equivalent to  $\mathbb{L}_{\mathbb{P}}^{U(t,\cdot)}$  on  $C(\mathbb{R}; \mathbb{R})$ , for every  $t \in [0, T]$ .*

By proving law equivalence between  $e_{heat}^{Neu}(a, b, h)$  and  $e_{heat}^{Neu}(a, b + d, h)$  under considerably weaker conditions, Theorem 1.1 and Corollary 1.2 extend and make more applicable the notion of relative absolute continuity in our earlier work (Theorem 3.3.3 in [2] or Theorem 4.3 in [3]). Like Theorem 4.3 in [3], Theorem 1.1 and Corollary 1.2 (and thus the first assertion of Theorem 1.2 below) are equally valid for wave SPDEs, space-time SDEs, and SDEs (cf. Theorem 3.7, Theorem 4.3, Theorem 5.3, and their proofs under the stronger conditions of [3]). An interesting application of Theorem 1.1 and Corollary 1.2 is to allow us to prove the following theorem about the compact support property of solutions to a large class of SPDEs containing the Allen-Cahn SPDE:

**Theorem 1.2.** *Assume that the conditions of Theorem 1.1 hold. Then,  $U(\cdot, \cdot)$  ( $U(t, \cdot)$ ) has compact support iff  $V(\cdot, \cdot)$  ( $V(t, \cdot)$ ) does. In particular, if  $V(t, x)$  is a solution to the Allen-Cahn SPDE (1.3),  $h(x)$  has compact support, and  $\frac{1}{2} \leq \gamma < 1$ ; then, for each  $t \in [0, T]$ ,  $V(t, \cdot)$  has compact support as a function of  $x$  iff (1.4) holds.*

It is noteworthy that all the Allen-Cahn SPDE results and their proofs here and in [5] are valid for the KPP SPDE, obtained by replacing the Allen-Cahn term  $2V(1 - V^2)$  by the KPP term  $V(1 - V)$ . In [2, 4], we gave a proof of the existence of solutions to heat SPDEs with continuous diffusion coefficient  $a$  and measurable drift  $b$ —with  $a$  satisfying a linear growth condition and  $b/a$  satisfying Novikov's condition—using a system of stochastic differential-difference equations (SDDEs). In [6], we use our SDDE approach and the results of this note and [5] to further investigate the existence and some properties of SPDEs considered here.

## 2. PROOFS OF RESULTS

Proof of Theorem 1.1. It follows from the uniqueness in law assumption for  $e_{heat}^{Neu}(a, b, h)$ , the almost sure  $L^2(\mathcal{R}_T, \lambda)$  condition on  $R_V$ , and a trivial extension of Theorem 1.1 in [5] to the space  $\mathcal{R}_T$  that we have uniqueness in law for  $e_{heat}^{Neu}(a, b + d, h)$  (see Remark 1.1).

Now, take  $\{\tau_n^U\}$  and  $\{\tau_n^V\}$  to be the sequences of stopping times

$$(2.1) \quad \tau_n^U \triangleq T \wedge \inf \left\{ 0 \leq t \leq T; \int_{[0, t] \times \mathbb{R}} R_U^2(s, x) ds dx = n \right\}; \quad n \in \mathbb{N},$$

and  $\{\tau_n^V\}$  is gotten from (2.1) by replacing  $U$  with  $V$ . Let  $\tilde{\mathcal{W}} = \{\tilde{\mathcal{W}}_t(B), \mathcal{F}_t; 0 \leq t \leq T, B \in \mathcal{B}(\mathbb{R})\}$  be given by

$$\tilde{\mathcal{W}}_t(B) \triangleq \mathcal{W}_t^{(2)}(B) - \int_{[0, t] \times B} R_U(s, x) ds dx.$$

Novikov's condition and Girsanov's theorem for white noise (see Corollary 3.1.3 in [2]) imply that, for  $n \in \mathbb{N}$ ,  $\tilde{\mathcal{W}}_n = \{\tilde{\mathcal{W}}_{t \wedge \tau_n^U}(B), \mathcal{F}_t; 0 \leq t \leq T, B \in \mathcal{B}(\mathbb{R})\}$  is a white noise stopped at time  $\tau_n^U$ , under the probability measure  $\tilde{\mathbb{P}}_n$  defined on  $\mathcal{F}_T$  by the recipe

$$(2.2) \quad \begin{aligned} \frac{d\tilde{\mathbb{P}}_n}{d\mathbb{P}} &= \Xi_{T \wedge \tau_n^U}^{R_U, \mathcal{W}}(\mathbb{R}) \\ &\triangleq \exp \left[ \int_{[0, T \wedge \tau_n^U] \times \mathbb{R}} R_U(s, x) \mathcal{W}^{(2)}(ds, dx) - \frac{1}{2} \int_{[0, T \wedge \tau_n^U] \times \mathbb{R}} R_U^2(s, x) ds dx \right]. \end{aligned}$$

It then follows that  $(U, \tilde{\mathcal{W}}_n), (\Omega^{(2)}, \mathcal{F}_T, \{\mathcal{F}_t\}, \tilde{\mathbb{P}}_n)$  is a solution to the  $e_{heat}^{Neu}(a, b + d, h)$  on  $\mathcal{R}_{T \wedge \tau_n^U} \triangleq [0, T \wedge \tau_n^U] \times \mathbb{R}$ , for each  $n \in \mathbb{N}$ . Consequently for an arbitrary set  $\Lambda \in \mathcal{B}(C(\mathcal{R}_T; \mathbb{R}))$  we get

$$(2.3) \quad \begin{aligned} \mathbb{Q}[V(\cdot, \cdot) \in \Lambda, \tau_n^V = T] &= \tilde{\mathbb{P}}_n[U(\cdot, \cdot) \in \Lambda, \tau_n^U = T] \\ &= \mathbb{E}_{\mathbb{P}} \left[ 1_{\{U(\cdot, \cdot) \in \Lambda, \tau_n^U = T\}} \Xi_{T \wedge \tau_n^U}^{R_U, \mathcal{W}}(\mathbb{R}) \right]; \quad n \in \mathbb{N}. \end{aligned}$$

To see (2.3) observe that, on the event  $\Omega_n^U \triangleq \{\omega \in \Omega^{(2)}; \tau_n^U(\omega) = T\}$ ,  $(U, \tilde{\mathcal{W}}_n)$  is a solution to  $e_{heat}^{Neu}(a, b + d, h)$  on  $\mathcal{R}_T$ , under  $\tilde{\mathbb{P}}_n$ , and so the uniqueness in law for  $e_{heat}^{Neu}(a, b + d, h)$  and the definitions of  $\tau_n^U$  and  $\tau_n^V$  give the first equality in (2.3). By the  $L^2$  assumption on  $R_V$  and the definition of  $\tau_n^V$ , we have  $\lim_{n \rightarrow \infty} \mathbb{Q}[\tau_n^V = T] = 1$  so that taking limits in (2.3) we get

$$(2.4) \quad \mathbb{Q}[V(\cdot, \cdot) \in \Lambda] = \lim_{n \rightarrow \infty} \tilde{\mathbb{P}}_n[U(\cdot, \cdot) \in \Lambda, \tau_n^U = T] = \lim_{n \rightarrow \infty} \mathbb{E}_{\mathbb{P}} \left[ 1_{\{U(\cdot, \cdot) \in \Lambda, \tau_n^U = T\}} \Xi_{T \wedge \tau_n^U}^{R_U, \mathcal{W}}(\mathbb{R}) \right].$$

Clearly, if  $\mathbb{P}[U(\cdot, \cdot) \in \Lambda] = 0$  then  $\mathbb{E}_{\mathbb{P}} \left[ 1_{\{U(\cdot, \cdot) \in \Lambda, \tau_n^U = T\}} \Xi_{T \wedge \tau_n^U}^{R_U, \mathcal{W}}(\mathbb{R}) \right] = 0$  for each  $n$ , and so

$$\mathbb{Q}[V(\cdot, \cdot) \in \Lambda] = \lim_{n \rightarrow \infty} \mathbb{E}_{\mathbb{P}} \left[ 1_{\{U(\cdot, \cdot) \in \Lambda, \tau_n^U = T\}} \Xi_{T \wedge \tau_n^U}^{R_U, \mathcal{W}}(\mathbb{R}) \right] = 0.$$

I.e.,  $\mathbb{L}_{\mathbb{Q}}^V$  is absolutely continuous with respect to  $\mathbb{L}_{\mathbb{P}}^U$  (on  $\mathcal{B}(C(\mathcal{R}_T; \mathbb{R}))$ ). A similar argument yields the absolute continuity of  $\mathbb{L}_{\mathbb{P}}^U$  with respect to  $\mathbb{L}_{\mathbb{Q}}^V$ , and we will omit

it.  $\square$

Our compact support result for the Allen-Cahn SPDE (1.3) can now be proved.

**Proof of Theorem 1.2.** To see the compact support transfer among (1.1) and (1.2), observe that if  $\mathbb{P}[U(\cdot, \cdot) \in C_c(\mathcal{R}_T; \mathbb{R})] = 1$  ( $\mathbb{P}[U(t, \cdot) \in C_c(\mathbb{R}; \mathbb{R})] = 1$ ), then by Theorem 1.1 and Corollary 1.2 we have  $\mathbb{Q}[V(\cdot, \cdot) \in C_c(\mathcal{R}_T; \mathbb{R})] = 1$  ( $\mathbb{Q}[V(t, \cdot) \in C_c(\mathbb{R}; \mathbb{R})] = 1$ ), respectively, and vice versa.

If  $\frac{1}{2} \leq \gamma < 1$  and the integrability condition (1.4) is satisfied by solutions of the Allen-Cahn SPDE (1.3), then by Corollary 1.1 the law of (1.3) is equivalent to that of the associated heat SPDE (without the Allen-Cahn nonlinearity). Now, observe that if  $U$  is a solution to the heat SPDE associated with (1.3); then by [10, 12] we have that, for each  $t \in [0, T]$ ,  $U(t, \cdot)$  has compact support (in the space variable) almost surely if  $h(x)$  has compact support and if  $0 < \gamma < 1$ . It then follows, as in the proof of the first part of Theorem 1.2 (the compact support transfer among (1.1) and (1.2)), that if  $V$  is a solution to the Allen-Cahn SPDE (1.3); then, for each  $t \in [0, T]$ ,  $V(t, \cdot)$  has compact support (in space) almost surely whenever  $h(x)$  is compactly supported and  $\frac{1}{2} \leq \gamma < 1$ . In the opposite direction, the compact supportedness of  $V(t, \cdot)$  for each  $t \in [0, T]$  trivially implies the integrability in (1.4) for  $\frac{1}{2} \leq \gamma < 1$ .  $\square$

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